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On Nonlinear Minimization Problems and Lf-Splines, I*

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1. INTRODUCTION

Let X, Y be a pair of separated locally convex spaces and let $T: X \to Y$ be a continuous operator. Given a subset C of X, an element $y \in Y$ and a lower semi-continuous (l.s.c.) proper convex function f on Y, we deal here with the problem of minimizing f over the translate $T(C) - y$ of the set $T(C)$. In case X, Y are Banach spaces, $f = \|\cdot\|_Y$ the norm of Y, T is linear and $y = \Theta$, this constitutes the so-called generalized spline problem which has been extensively investigated recently (cf., e.g., [1,5–8]). A comprehensive account of this problem along with related applications may be found in [2].

In [3] and [4], substituting a norm by a l.s.c. sublinear functional defined on a separated locally convex space, many standard results of the theory of best approximation in normed spaces have been extended. In the present exposition, we treat in the same spirit the abstract minimization problem of [2] in an extended framework. This leads us to define in Section 3 the so-called Lf-spline.

We adopt the standard framework of convex analysis as employed in [9, Chapter 6]. Let Y, V be a pair of linear spaces put in duality by a separating bilinear form $\langle\,.\,\rangle$. A topology on Y is said to be compatible with the pairing if it is a separated locally convex topology for which continuous linear functionals on Y are precisely those of the form

$$\langle\,\cdot\,, v\rangle : y \to \langle y, v\rangle, \qquad \text{for} \quad v \in V.$$

Likewise, a compatible topology on V is a separated locally convex topology for which continuous linear functionals on V are precisely those of the form

$$\langle y, \cdot\,\rangle : v \to \langle y, v\rangle, \qquad \text{for} \quad y \in Y.$$

Let Y, V be equipped with compatible topologies. By a proper convex function on Y, we mean a function $f: Y \to \mathbb{R} \cup \{+\infty\}$ which is convex and $\not\equiv +\infty$.

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2. A General Nonlinear Minimization Problem

Let (X, U), (Y, V) be pairs of real linear spaces put in duality by separating bilinear forms $\langle\,,\rangle_{X\times U}$, $\langle\,,\rangle_{Y\times V}$ resp. and equipped with compatible locally convex topologies. Let us be given a l.s.c. proper convex function f on Y, a continuous operator T: $X \to Y$ and a non-empty closed subset C of X. Given an element $y \in Y$, one is concerned here with the minimization problem

$$(P_y) \qquad \alpha = \min_{x \in C} f(Tx - y).$$

The following definitions are pertinent for studying existence of solutions of problem (P_y).

Definition 2.1. (i) C is said to be *P_y-compact* (resp. *P_y-b-compact*) if each (resp. each bounded) minimizing net $\{x_\lambda\}$ for (P_y) (i.e., a net (resp. a bounded net) $\{x_\lambda\}$ in C satisfying $f(Tx_\lambda - y) \to \alpha$) has a convergent subnet. C is said to be *P-compact* (resp. *P-b-compact*) if C is P_y-compact (resp. P_y-b-compact) for each $y \in Y$.

(ii) T is said to satisfy *property* (H) (resp. *property* (H_b)) if each (resp. bounded) net $\{x_\lambda\}$ in X for which $\{Tx_\lambda\}$ is convergent in Y has a convergent subnet. (In these definitions, whenever we speak of a bounded net, it is to be understood as bounded with respect to *some* compatible topology of X, not necessarily the initial one.) Recall that (cf. [9, p. 347]) f is said to be *inf-compact* if the sublevel sets $S_\lambda := \{y \in Y: f(y) \leqslant \lambda\}$ of f are compact for each $\lambda \in \mathbb{R}$, and the set $T(C)$ is said to be f-bddly compact if $(T(C) - y) \cap S_\lambda$ is compact for each $y \in Y$ and $\lambda \in \mathbb{R}$. (Note that (cf. [3, p. 458]), in case f is sub-linear and $T(C)$ is closed, this is equivalent to $T(C) \cap S_\lambda$ is compact for each $\lambda \in \mathbb{R}$.)

We remark that if Y is a normed space with the norm topology (resp. weak topology) and $f = \|\cdot\|_Y$, then f is inf-compact if and only if Y is finite dimensional (resp. reflexive). Also, if Y is the normed dual of V, then $f = \|\cdot\|_Y$ is $\sigma(Y, V)$-inf-compact. More generally, by a theorem of Moreau (cf. [9, p. 347]), f is $\sigma(Y, V)$ inf-compact iff f^* is finite and $\tau(V, Y)$-continuous at Θ. Here $\tau(V, Y)$ denotes the Mackey topology of V.

Proposition 2.2. *Assume that $T(C)$ is closed. Consider the following statements*:

(i) *f is* inf-*compact*;

(ii) *$T(C)$ is f-bddly compact*;

(iii) *the problem (P_y) has a solution for each $y \in Y$. One has* (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

Proof. This follows immediately from the definitions. ∎

PROPOSITION 2.3. *Assume that*

(i) *$T(C)$ is f-bddly compact, and*

(ii) *T satisfies (H) (resp.*

(ii′) *T satisfies (H_b) and C contains a bounded minimizing net for (P_y) for each $y \in Y$).*

Then C is P-compact (resp. P-b-compact).

Proof. Let $\{x_\lambda\}$ be a minimizing (resp. bounded minimizing) net for problem (P_y). Fix up $\bar{\lambda}$ and consider the set

$$T(C) \cap \{y + S_{f(Tx_{\bar{\lambda}} - y)}\},$$

which is compact, in view of (i). Thus the net $\{Tx_\lambda\}$, which eventually lies in this set, has a convergent subnet. In view of (H) (resp. (H_b)), $\{x_\lambda\}$ has a convergent subnet and C is P_y-compact (resp. P_y-b-compact). ∎

Remark. It is usually convenient in the preceding proposition to assume in place of (i), (i′): f is inf-compact. For example, in case Y is a Banach space and $f = \|\cdot\|_Y$, the most convenient hypothesis is that Y be reflexive.

PROPOSITION 2.4. *Assume that*

(i) *C is closed, and*

(ii) *C is P_y-compact (resp.*

(ii′) *C is P_y-b-compact and contains a bounded minimizing net for (P_y)).*

Then problem (P_y) has a solution.

Proof. Let $\{x_\lambda\}$ be a minimizing (resp. bounded minimizing) net for (P_y). Then C being P_y-compact (resp. P_y-b-compact), there is a convergent subnet of $\{x_\lambda\}$, which we denote again by $\{x_\lambda\}$, converging to $\bar{x}$. C being closed, $\bar{x} \in C$. Lower semi-continuity of f and continuity of T yield

$$f(T\bar{x} - y) \leqslant \liminf_{\lambda} f(Tx_\lambda - y) = \alpha.$$

Thus $\bar{x}$ is a solution of (P_y). ∎

THEOREM 2.5. *Assume that*

(i) *C is closed,*

(ii) *f is* inf-*compact, and*

(iii) *either* (ii) *or* (ii′) *of Proposition* 2.3 *holds.*

Then problem (P_y) has a solution for each $y \in Y$.

Proof. This follows immediately from Propositions 2.3 and 2.4. ∎

We remark that the preceding theorem holds if we assume T to be continuous with a compatible topology on Y *stronger* than the initial one.

COROLLARY 2.6. *Let X be a Banach space which is the normed dual of U and equipped with the topology $\sigma(X, U)$. Let C be a $\sigma(X, U)$-closed subset of X and let T: $X \to Y$ be continuous. Let f be a l.s.c.* inf-*compact proper convex function on Y. Then problem (P_y) has a solution, provided there is a norm-bounded minimizing net for this.*

Proof. By the Banach–Alaoglu theorem, T satisfies property (H_b) for norm-bounded minimizing nets for (P_y) and the corollary follows from the preceding theorem. ∎

EXAMPLE 2.7. Let X be the Sobolev space $W^{q,p}(0, 1)$, $Y = L^p(0, 1)$, $1 < p < \infty$. Let ϕ be a $\mathscr{C}^{(1)}$ function on $[0, 1] \times \mathbb{R}^q$. Let C be defined by

$$C = \{x \in X: \Lambda_i x = \lambda_i, i = 1, 2, ..., q\},$$

where Λ_i, $1 \leqslant i \leqslant q$, are prescribed continuous linear functionals on $\mathscr{C}^{(q-1)}[0, 1]$ which are linearly independent over $\mathscr{P}_{q-1}$:= polynomials of degree $\leqslant q - 1$, and λ_i's are prescribed real numbers. Let T: $X \to Y$ be a nonlinear differential operator defined by

$$Tx(t) = x^{(q)}(t) + \phi(t, x(t), ..., x^{(q-1)}(t)).$$

Let $\{f_j: 1 \leqslant j \leqslant m\}$ be a prescribed set of l.s.c. proper convex functions on Y and set

$$f(y) = \max\{\|y\|_p, \max_{1 \leqslant j \leqslant m} f_j(y)\}.$$

Let X, Y be equipped with weak topologies. Since $S_\lambda \subset \{y \in Y: \|y\|_p \leqslant \lambda\}$, the latter set being weakly compact, f is inf-compact. Since D^q is weakly continuous and since weak convergence in X implies uniform convergence of derivatives through order $q - 1$ on $[0, 1]$, T is continuous for these topologies. Furthermore, if we assume ϕ to be bounded on $\mathbb{R}^q$, then it is easily verified that a norm bounded minimizing sequence exists for problem (P_y) and that (H_b) is satisfied by T (X being reflexive). Hence all conditions of Theorem 2.5 are fulfilled and problem (P_y) has a solution.

For $p = \infty$, let C be a weak*-closed subset of $W^{q,\infty}(0, 1)$. Take the compatible topologies of X, Y as the weak*-topologies. Proceeding exactly as before, an application of Corollary 2.6 yields existence of solution of problem (P_y).

3. Linear Minimization Problems

In what follows, we adopt the same framework as in Section 2 and assume throughout this section that $T = L\colon X \to Y$ is a bounded linear operator. As in Section 1, let C be a closed subset of X and f be a l.s.c. proper convex function on Y. A solution of problem (P_Θ) (with $y = \Theta$) will be called an *Lf-spline in C*. We shall mainly dwell here on existence results for *Lf*-splines which are based on Proposition 2.2 and Theorem 2.5.

For employing Proposition 2.2, it is necessary to check that $A = L(C)$ is closed. Assume X, Y to be Fréchet spaces for some compatible topologies (not necessarily the initial ones). Let $L\colon X \to Y$ be a bounded linear operator with closed range (for Fréchet topology). Let $\mathscr{N}(L) :=$ the kernel of L. Then A is closed if and only if $L^{-1}(A) = \mathscr{N}(L) + C$ is closed in X. Indeed, this follows from the fact that in the diagram

$$\begin{array}{ccc} X & \xrightarrow{L} & Y \\ {\scriptstyle \eta}\big\downarrow & \nearrow_{\tilde{L}} & \\ X/\mathscr{N}(L) & & \end{array} \qquad (\tilde{L}[x] = Lx)$$

the canonical surjection η is an open mapping and that $\tilde{L}$ is a topological isomorphism by the open mapping theorem. If we assume C to be convex and $\mathscr{N}(L) + C$ to be closed in X, then A is closed convex for the Fréchet topology of Y, and hence it is closed for the initial compatible topology of Y. We need the following refinement of a result of Dieudonné due to Joly (cf. Laurent [9, p. 490]): *Let A and B be two non-empty closed convex subsets of a separated linear topological space, such that A is locally compact and $A_\infty \cap B_\infty$ is a linear subspace (here A_∞ denotes the asymptotic or recession cone of A, $A_\infty := \bigcap_{\lambda > 0} \lambda(A - a)$, for any $a \in A$), then $A + B$ is closed.*

Theorem 3.1. *Let X, Y be Fréchet spaces for some compatible topologies (not necessarily the initial ones). Let $L\colon X \to Y$ be a bounded linear operator with closed range (for the Fréchet topologies) and with a finite-dimensional kernel. Assume C to be closed convex subset of X such that $C_\infty \cap \mathscr{N}(L)$ is a linear subspace. Let f be a l.s.c.,* inf*-compact, proper convex function on Y (for the initial compatible topology of Y). Then problem (P_y) has a solution for each $y \in Y$; in particular, there exists an Lf-spline in C.*

Proof. This follows immediately from the preceding discussion by an application of Proposition 2.2. ∎

Remarks. (i) The preceding theorem subsumes most of the known existence theorems for splines (e.g., Theorem 9.3.3 of Laurent [9], Theorem 2.1 of Jerome and Varga [6] for Lg-splines, Theorem of Holmes [5] for R-splines).

(ii) Frequently, in applications of Theorem 3.1, the convex set C occurs either as an intersection of half-space constraints,

$$C = \{x \in X: \langle x, u_i \rangle \leqslant r_i, i \in I\},$$

$\{r_i\}_{i \in I}$ and $\{u_i\}_{i \in I}$ being prescribed subsets of $\mathbb{R}$ and U, respectively, or it occurs as

$$C = \{x \in X: R_i x \in K_i, i \in I\},$$

where R_i: $X \to Z_i$, $i \in I$, are continuous linear maps, $\{Z_i\}_{i \in I}$ being a prescribed family of locally convex spaces and $K_i \subset Z_i$ being closed convex sets. In the former case, $C_\infty = C_0$, where $C_0 = \{x \in X: \langle x, u_i \rangle \leqslant 0, i \in I\}$, and in the latter case C_∞ is contained in any linear subspace containing C. Hence, for Λ to be closed, it suffices to assume in the first case that $C_0 \cap \mathcal{N}(L)$ is a linear subspace, and in the second case that C is contained in the algebraic complement of $\mathcal{N}(L)$ (recall that we are assuming $\mathcal{N}(L)$ to be finite dimensional).

Proposition 3.2. *Let X, Y be Fréchet spaces for compatible topologies stronger than the initial ones. Let L: $X \to Y$ be a linear operator, continuous with respect to the Fréchet topology of X and the Fréchet topology of Y. Assume that L has a finite dimensional kernel and a closed range (for the Fréchet topology of Y). Then L satisfies property (H_b) for the initial topologies.*

Proof. The proof is patterned after the proof of Theorem 2.1 of [2]. Since L has a closed range, we may assume L to be surjective. Let $\{x_\lambda\}$ be a bounded net in X (for the Fréchet topology of X) with $\{Lx_\lambda\}$ convergent to Lx_0 in Y for the initial topology. Since $N = \mathcal{N}(L) :=$ kernel of L, is finite dimensional, it is complemented in X. Let the closed subspace M be the complement of N, i.e., $X = N \oplus M$. Let $\tilde{L} = L|_M$. By the open mapping theorem, $\tilde{L}$ is a topological isomorphism of M onto Y (for Fréchet topologies). Let $x_\lambda = m_\lambda + n_\lambda$ with $m_\lambda \in M$ and $n_\lambda \in N$. Then $\tilde{L}m_\lambda \to \tilde{L}m_0$ for the initial topology of Y. Hence, $m_\lambda \to m_0$ for the initial topology of X. Also, $\{n_\lambda\}$ being a bounded net in N which is finite dimensional, we may assume $n_\lambda \to n_0$ in N for the Fréchet and hence for the weaker initial topology of X. Thus, $x_\lambda \to n_0 + m_0$ (initial topology) and the proof is complete.

Theorem 3.3. *Let X, Y be Fréchet spaces with compatible topologies stronger than the initial ones. Let L: $X \to Y$ be a linear operator with closed* *range, continuous with respect to the Fréchet topology of X and the Fréchet topology of Y, and satisfying property (H_b) (bounded nets in X are to be understood in the Fréchet topology). Let $R\colon X \to Z$ be a surjective continuous linear operator (with Fréchet topology of X), Z being a Fréchet space, and let K be a closed bounded subset of Z. Let*

$$C = \{x \in X \colon Rx \in K\}.$$

Let f be a l.s.c., inf-*compact, proper convex function on Y for the initial topology. Then, for each $y \in Y$, problem (P_y) has a solution; in particular, there exists an Lf-spline in C.*

Proof. In view of Theorem 2.5, it suffices to produce a bounded minimizing net for problem (P_y). Let $\{x_\lambda\}$ be a minimizing net for (P_y). Since f is inf-compact, we may assume, without loss of generality, that $\{Lx_\lambda\}$ is convergent in Y, for the initial topology of Y, by choosing an appropriate subnet. From this one concludes that $\{|x_\lambda|\}$ is bounded in $X/\mathcal{N}(L)$, and hence $\{x_\lambda + x'_\lambda\}$ is bounded in X for an appropriate net $\{x'_\lambda\}$ in $\mathcal{N}(L)$. $\{R(x_\lambda + x'_\lambda)\}$ and $\{Rx_\lambda\}$ both being bounded nets in Z, so also is the net $\{Rx'_\lambda\}$. Using the quotient space $\mathcal{N}(L)/\mathcal{N}(R)$, one obtains a net $\{x''_\lambda\} \subset \mathcal{N}(L) \cap \mathcal{N}(R)$ such that $\{x'_\lambda + x''_\lambda\}$ is bounded, whence it follows that the net $\{x_\lambda - x''_\lambda\}$ is bounded and is contained in C. Moreover, $L(x_\lambda - x''_\lambda) = Lx_\lambda$. Hence, $\{x_\lambda - x''_\lambda\}$ is a bounded minimizing net. ∎

Remark. In the preceding theorem, assume either L to have a finite-dimensional kernel *or* X to be a reflexive Banach space with weak topology as its initial compatible topology *or* X to be a Banach space which is normed dual of U with topology $\sigma(X, U)$ for its initial topology. Then property (H_b) is easily seen to hold.

EXAMPLE 3.4. Let $X = W^{q,p}(a, b)$, $1 < p \leqslant \infty$, and let $Y = L^p(a, b)$. Let $L = D^q + \sum_{j=0}^{q-1} a_j(t) D^j$, where $a_j \in \mathscr{C}[a, b]$, $0 \leqslant j \leqslant q-1$. Let $\{\Lambda_j\colon 1 \leqslant j \leqslant n\}$ be a linearly independent set of elements of X^* and define $R\colon X \to \mathbb{R}^n$ by $Rx = (\Lambda_1 x, \ldots, \Lambda_n x)$. Let K be a compact, convex subset of $\mathbb{R}^n$ and set $C = \{x \in X\colon Rx \in K\}$. Let f be as in Example 2.7,

$$f(y) = \max\{\|y\|_p, \max_{1 \leqslant j \leqslant m} f_j(y)\} \qquad (y \in Y).$$

For $1 < p < \infty$, take weak topologies as the initial compatible topologies for X and Y. Then f is inf-compact. By the preceding remark and the last theorem, an Lf-spline exists in C. For $p = \infty$, we observe that $W_0^{q,\infty}(a, b)$ is the dual of $W_0^{q,1}(a, b)$. Hence, $\mathscr{P}_{q-1} \oplus W_0^{q,\infty}(a, b)$, which is equivalent to $W^{q,\infty}(a, b)$, is a normed dual of a separable Banach space. Hence, by the preceding remark and the last theorem again, an Lf-spline exists in C.

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